Electrostatic and Magnetostatic Solutions in a Lorentz-Violating Electrodynamics Model

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Abstract

We propose an effective Lorentz violating electrodynamical model via static de Sitter metric which is deviated from Minkowski metric by a minuscule amount depending on the cosmological constant. We obtain the electromagnetic field equations via the vierbein decomposition of the tensors. In addition, as an application of the electromagnetic field equations obtained, we get the solutions of electrostatic field and magnetostatic field due to a point charge and a circle current respectively and discussed the implication of the effect of Lorentz violation in our electromagnetic theory.

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Lorentz invariance is one of the most great discoveries of physics in the history of physics and has been confirmed to ever greater precision. Most of the evidence comes from short distance tests. However there are plenty of signs that something strange may happen at large distance (for example,dark energy), where the constraints on Lorentz symmetry are much weaker. It is reasonable that many researchers are interesting in Lorentz violation(LV) from various points of views [1, 2, 3, 4, 5, 6]. They pointed out that Lorentz invariance can be viewed as a low energy effective invariance. Remarkably, under suitable circumstances, some experimental information about quantum gravity can nonetheless be obtained. The point is that minuscule effects emerging from the underlying quantum gravity might be detected in sufficiently sensitive experiments. To be identified as definitive signals from the Planck scale, such effects would need to violate some established principle of low-energy physics. One promising class of potential effects is relativity violations, arising from breaking the Lorentz symmetry that lies at the heart of relativity. Recent proposals suggest LV effects could emerge from strings, loop quantum gravity, noncommutative field theories, or numerous other sources at the Planck scale[7]. On the other hand, recent observations, such as the luminosity observations of the farthest supernovas [8], show that our universe is accelerated expanding , asymptotic de Sitter with a positive cosmological constant Λ [9, 10, 11].

Among the developments on LV research, one is a systematic extension of the standard model of particle physics incorporating all possible LV in the renormalizable sector called Standard Model Extension(SME), developed by Colladay and Kostelecký [12]. That provided a framework for computing in effective field theory the observable consequences for many experiments and led to much experimental work setting limits on the LV parameters in the Lagrangian [13].

We limit our attention in the present work to the sector of classical Lorentz violating electrodynamics in Minkowski space-time , coupled to an arbitrary 4-current source, in this framework of SME. Due to the presence of dark energy or the nonzero positive cosmological constant, the space-time without any matter is de Sitter rather than Minkowskian and so it is a natural way to substitute the Lorentz invariant low energy effective theory with its covariant formulation in de Sitter space-time for a field theory in Minkowski space-time. The Lorentz invariance is violated obviously in this way to the observer in Minkowski space-time. We mean in this approach that the Lorentz symmetry is an approximately symmetry of the low energy effective theory. There are different types of metric for de Sitter space-time

and it is well known that quantum field theory in de Sitter space-time equipped with the static metric is a finite temperature field theory in a pure field theory in curved space-time approach[14]. However, we investigate the Lorentz violating electrodynamics in Minkowski space-time in the present works and so there is no finite temperature problem here. In this way, different choices of de Sitter metrics to formulate the Lorentz violating theory is just following different scenarios to violate Lorentz symmetry. In this paper we propose a scenario that the substitute of Lorentz invariant electromagnetic theory is its formulation in de Sitter space-time equipped with the static metric which is obviously Lorentz violating. We define all the observables in Minkowski space-time as the corresponding vierbein decomposition components in de Sitter space-time of corresponding physical quantities.

We now set up the LV electrodynamics by the vierbein formalism. A basic object in the formalism is the vierbeins ϑ^a_μ, which can be viewed as providing at each point on the space-time manifold a link between the covariant components $T_{\lambda\mu\nu\ldots}$ of a tensor field in a coordinate basis and the corresponding covariant components $T_{abc\ldots}$ of the tensor field in a local Lorentz frame. The link is given by

$$T_{\lambda\mu\nu\ldots} = \vartheta^a{}_\lambda \vartheta^b{}_\mu \vartheta^a{}_\nu \ldots T_{abc\ldots} \tag{1}$$

In the coordinate basis, the components of the space-time metric are denoted $g_{\mu\nu}$. In the local Lorentz frame, the metric components take the Minkowski form $\eta_{ab} = diag(1, -1, -1, -1)$.

Then we'd like to introduce the de Sitter space and its metric, de Sitter space can be regarded as a 4-d hyperboloid S_R embedded in a 5-d Minkowski space with $\eta_{AB} = diag(1, -1, -1, -1, -1)$,

$$\begin{aligned} S_R: \quad \eta_{AB}\xi^A\xi^B &= -R^2, \\ ds^2 &= \eta_{AB} d\xi^A d\xi^B, \end{aligned} \tag{2}$$

where $A, B = 0, ..., 4$. Clearly, Eqs. (2) are invariant under de Sitter group $SO(1, 4)$. The metric of this space-time can be written as [15]

$$ds^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu - \frac{K(\eta_{\mu\nu}\xi^\mu d\xi^\nu)^2}{1 + K(\eta_{\mu\nu}\xi^\mu\xi^\nu)} \tag{3}$$

where $\mu, \nu = 0, ..., 3$, $K = \frac{1}{R^2} = \frac{\Lambda}{3}$, Λ is the cosmological constant. This metric is invariant under two classes of simple transformations (see, for example, P 387 of the book [15]), one is $SO(1, 3)$ transformations:

$$\xi'^\mu = L^\mu{}_\nu \xi^\nu \tag{4}$$

the other is the 'quasitranslations', with

$$\xi'^{\mu} = \xi^{\mu} + a^{\mu}[(1 - K\eta_{\rho\sigma}\xi^{\rho}\xi^{\sigma})^{1/2} - bK\eta_{\rho\sigma}\xi^{\rho}a^{\sigma}] \qquad (5)$$
$$b = \frac{1 - (1 - K\eta_{\rho\sigma}a^{\rho}a^{\sigma})^{1/2}}{K\eta_{\rho\sigma}a^{\rho}a^{\sigma}}$$

In particular, these transformations take the origin $\xi^{\mu} = 0$ into any a^{μ}. For the metric given by Eq. (3), we can introduce coordinates in which the metric appears time-independent by

$$x^i = \xi^i = x'^{\,i} \exp(K^{1/2}t'), \qquad (6)$$
$$\xi^0 = \frac{1}{\sqrt{K}}[\frac{K\mathbf{x}'^2}{2}\cosh(K^{1/2}t') + (1 + \frac{K\mathbf{x}'^2}{2})\sinh(K^{1/2}t')],$$
$$t = t' - \frac{1}{2K^{1/2}}\ln[1 - K\mathbf{x}'^2\exp(2K^{1/2}t')].$$

Then Eq. (3) becomes

$$ds^2 = (1 - K\mathbf{x}^2)dt^2 - d\mathbf{x}^2 - \frac{K(\mathbf{x}\cdot d\mathbf{x})^2}{1 - K\mathbf{x}^2}. \qquad (7)$$

It can be showed that the spatial metric of the space-time is just the metric of a 3-d spherical surface (with radius R) in 4-d Euclidean space. However, unlike the metric given by Eq. (3), this static de Sitter metric is obviously Lorentz violating. Noting that the transformation (5) and (6) leaving the metric (7) invariant also take the spatial origin $\mathbf{x} = \mathbf{0}$ into any $\mathbf{a}$ and contain the spatial $SO(3)$ rotation. Choosing the spherical coordinate, we can rewrite the static metric Eq. (7) as follow

$$ds^2 = \sigma dt^2 - \frac{1}{\sigma}dr^2 - r^2 d\theta^2 - r^2\sin\theta^2 d\phi^2, \qquad (8)$$

where $\sigma = 1 - Kr^2$. So we can define a local Lorentz frame with vierbeins $\vartheta^a_{\mu}, a = 0, 1, 2, 3,$ where

$$\vartheta^a_{\mu} = diag(\sqrt{\sigma}, \frac{1}{\sqrt{\sigma}}, r, r\sin\theta). \qquad (9)$$

As in general relativity, the observables are vectors and tensors in the local Lorentz frame. Here we define the observables in Minkowski space-time as the vierbein decomposition components of the corresponding tensors of physical quantities in de Sitter space-time. In the present work, what we concerned is that the observables of electromagnetic field, the electric field strength $\overrightarrow{\mathbf{E}}$ and magnetic field strength $\overrightarrow{\mathbf{B}}$.

First, we introduce the electromagnetic potential contravariant vector

$$A^\mu = e^\mu_a A^a = (\frac{1}{\sqrt{\sigma}}\varphi, \sqrt{\sigma}A_r, \frac{1}{r}A_\theta, \frac{1}{r\sin\theta}A_\phi) \tag{10}$$

where

$$e^\mu_a = \eta_{ab}g^{\mu\nu}\vartheta^b_\nu = diag(\frac{1}{\sqrt{\sigma}}, \sqrt{\sigma}, \frac{1}{r}, \frac{1}{r\sin\theta}), A^a = (\varphi, A_r, A_\theta, A_\phi)$$

and A^a are components of the 'ordinary' vector [15]. This vector is that what we are seeking for, i.e. the observable vector. So the covariant 1-form can be written as below(we define $x^\mu = (t, r, \theta, \phi)$ hereafter),

$$A = A_\mu dx^\mu = \sqrt{\sigma}\varphi dt - \frac{1}{\sqrt{\sigma}}A_r dr - rA_\theta d\theta - r\sin\theta A_\phi d\phi. \tag{11}$$

Then, we can introduce the electromagnetic field-strength covariant 2-form $F = dA = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. Accordingly, here F_{ab} are components of the 'ordinary' electromagnetic field strength tensor and they can be written down as a matrix:

$$F_{ab} = \begin{pmatrix} 0 & -E_r & -E_\theta & -E_\phi \\ E_r & 0 & B_\phi & -B_\theta \\ E_\theta & -B_\phi & 0 & B_r \\ E_\phi & B_\theta & -B_r & 0 \end{pmatrix} \tag{12}$$

Then $F_{\mu\nu}$ becomes

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_r & -r\sqrt{\sigma}E_\theta & -r\sin\theta\sqrt{\sigma}E_\phi \\ E_r & 0 & \frac{r}{\sqrt{\sigma}}B_\phi & -\frac{r\sin\theta}{\sqrt{\sigma}}B_\theta \\ r\sqrt{\sigma}E_\theta & -\frac{r}{\sqrt{\sigma}}B_\phi & 0 & r^2\sin\theta B_r \\ r\sin\theta\sqrt{\sigma}E_\phi & \frac{r\sin\theta}{\sqrt{\sigma}}B_\theta & -r^2\sin\theta B_r & 0 \end{pmatrix}. \tag{13}$$

The action of the electromagenetic field can be writen as

$$I_M = \int(-F\wedge *F - A\wedge *j). \tag{14}$$

Thereinto, the symbol '$*$' is the Hodge-dual operator. As doing with the familiar formulae for gradient, curl, and divergence in the classical curvilinear coordinate systems, we now introduce these things in the spatial part of the local Lorentz frame, or accurately, on the submanifold S^3 of static de Sitter space-time manifold, by

$$\tilde{\nabla}\,\psi = d'\psi = \sqrt{\sigma}\psi,_r\vartheta^1 + \frac{1}{r}\psi,_\theta\vartheta^2 + \frac{1}{r\sin\theta}\psi,_\phi\vartheta^3, \tag{15}$$

$$\tilde{\nabla}\cdot\vec{f} = *'d'*'f = \frac{2}{r}\sqrt{\sigma}f_r + \sqrt{\sigma}f_{r,\,r} + \frac{\cos\theta}{r\sin\theta}f_\theta + \frac{1}{r}f_{\theta,\,\theta} + \frac{1}{r\sin\theta}f_{\phi,\,\phi}, \tag{16}$$

$$\begin{aligned}\tilde{\nabla}\times\vec{f} = *'d'f = (&\frac{\cos\theta}{r\sin\theta}f_\phi + \frac{1}{r}f_{\phi,\,\theta} - \frac{1}{r\sin\theta}f_{\theta,\,\phi})\vartheta^1 \\ &+(\frac{1}{r\sin\theta}f_{r,\,\phi} - \sqrt{\sigma}f_{\phi,\,r} - \frac{\sqrt{\sigma}}{r}f_\phi)\vartheta^2 \\ &+(\frac{\sqrt{\sigma}}{r}f_\theta + \sqrt{\sigma}f_{\theta,\,r} - \frac{1}{r}f_{r,\,\theta})\vartheta^3,\end{aligned} \tag{17}$$

$$\tilde{\nabla}^2\psi = (d'\delta' + \delta'd')\psi = \frac{\sqrt{\sigma}}{r^2}\frac{\partial}{\partial r}(r^2\sqrt{\sigma}\psi_{,\,r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\psi_{,\,\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}, \tag{18}$$

where $\vec{f}$ is an 'ordinary' 3-d vector on S^3 and $\vec{f} = f_r\vartheta^1 + f_\theta\vartheta^2 + f_\phi\vartheta^3 = (f_r, f_\theta, f_\phi)$. The independent basis $\vartheta^i = \vartheta^i_\mu dx^\mu,\ i = 1, 2, 3$ point out the orient of a 3-d vector in the local Lorentz frame. We use the symbol tilde and prime to indicate that we do these things on the submanifold S^3.

Noticing that Eq.(11)-(13) build a bridge from A^a to $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{B}}$, we can write down these relational equations to show it clearly, by performing some elementary calculations,

$$\vec{E} = -\frac{1}{\sqrt{\sigma}}\tilde{\nabla}(\sqrt{\sigma}\varphi) - \frac{1}{\sqrt{\sigma}}\frac{\partial\vec{A}}{\partial t} \tag{19}$$

and

$$\vec{B} = \tilde{\nabla}\times\vec{A} \tag{20}$$

where $\vec{A} = (A_r, A_\theta, A_\phi)$, $\vec{E} = (E_r, E_\theta, E_\phi)$ and $\vec{B} = (B_r, B_\theta, B_\phi)$, respectively.

Noticing that $F = dA$ is an exact 2-form, we can immediately obtain the Bianchi identity $dF = d^2A \equiv 0$ and the dynamical equation $\delta F = *d*F = j = j_\mu dx^\mu$, where we define

$$j^\mu = \vartheta^\mu_a j^a = (\frac{1}{\sqrt{\sigma}}\rho, \sqrt{\sigma}j_r, \frac{1}{r}j_\theta, \frac{1}{r\sin\theta}j_\phi) \tag{21}$$

with $j_a = (\rho, j_r, j_\theta, j_\phi)$. The 'ordinary' electric current density can be defined as before $\vec{j} = (j_r, j_\theta, j_\phi)$. Then the electromagnetic field equations in static de Sitter space-time are easy to obtained as bellow

$$\tilde{\nabla}\cdot\vec{B} = 0 \tag{22}$$

$$\tilde{\nabla}\times(\sqrt{\sigma}\vec{E}) + \frac{\partial\vec{B}}{\partial t} = 0 \tag{23}$$

$$\tilde{\nabla} \cdot \vec{E} = \rho \tag{24}$$

$$\tilde{\nabla} \times \vec{B} - \frac{1}{\sqrt{\sigma}} \frac{\partial \vec{E}}{\partial t} = \vec{j} \tag{25}$$

In addition, we study the covariant gauge condition of the electromagnetic field in static de Sitter space-time. In static de Sitter space-time, the reasonable gauge condition is the de Sitter covariant gauge condition $\delta A = 0$, in the local Lorentz frame, one can write this equation as follows:

$$\tilde{\nabla} \cdot (\sqrt{\sigma} \vec{A}) + \frac{\partial \varphi}{\partial t} = 0 \tag{26}$$

This de Sitter gauge will play an important role in dealing with the magnetostatic field.

Now we would like to investigate the interaction between the electromagnetic field and a charged source j^μ in static de Sitter space-time, as we do in Lorentz invariant electrodynamics, we write the Lagrange density of the system reads

$$\mathcal{L}_{\mathcal{M}} = \mathcal{L}_E + \mathcal{L'}_{\mathcal{M}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j_\mu A^\mu \tag{27}$$

here $\mathcal{L}_E = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ is the purely electromagnetic term and another term $\mathcal{L}_{\mathcal{M}}$ describes the charged particles (with charge e) and their electromagnetic interactions. Then the electromagnetic force $f^\mu(x)$ can be obtained as

$$f^\mu = F^\mu{}_\gamma j^\gamma = (e \frac{\vec{v} \cdot \vec{E}}{\sqrt{\sigma}}, \sqrt{\sigma} f_r, \frac{1}{r} f_\theta, \frac{1}{r \sin\theta} f_\phi) \tag{28}$$

with

$$\vec{f} = (f_r, f_\theta, f_\phi) = e(\vec{E} + \vec{v} \times \vec{B}).$$

If $K \to 0$, the LV electrodynamics tends to Lorentz invariant one and the force returned to Lorentz force. If we defined the purely electromagnetic term of energy-momentum tensor as usual,

$$T^{\alpha\beta}_{em} \equiv F^\alpha{}_\gamma F^{\alpha\beta} - \frac{1}{4} g^{\alpha\beta} F_{\lambda\delta} F^{\lambda\delta}, \tag{29}$$

we can obtain the energy-momentum conservation law for LV electrodynamics.

$$T^{\alpha\beta}_{em\,;\,\beta} = -F^\alpha{}_\beta j^\beta = -f^\alpha. \tag{30}$$

The symbol semicolon is the abbreviation for covariant derivative. To make this tensor equation familiar to us and have obviously observable meaning, we should rewrite the equation using the vierbein formalism as

$$\frac{1}{\sqrt{\sigma}} \tilde{\nabla} \cdot \vec{S} + \frac{1}{\sqrt{\sigma}} \frac{\partial \omega}{\partial t} = -\vec{j} \cdot \vec{E}, \tag{31}$$

$$\vec{S} = \sigma(\vec{E} \times \vec{B}), \omega = \frac{1}{2}(E^2 + B^2),$$

$$\tilde{\nabla} \cdot \overrightarrow{\overrightarrow{\mathcal{J}}} + \frac{1}{\sqrt{\sigma}} \frac{\partial \vec{g}}{\partial t} - K(\vec{r} \times \vec{E} \times \vec{E}) = -\vec{f}, \tag{32}$$

and

$$\overrightarrow{\overrightarrow{\mathcal{J}}} = -\vec{E}\vec{E} - \vec{B}\vec{B} + \frac{1}{2} \overrightarrow{\overrightarrow{\mathcal{I}}} (E^2 + B^2), \vec{g} = \vec{E} \times \vec{B},$$

where $\vec{S}$, ω, $\overrightarrow{\overrightarrow{\mathcal{J}}}$, and $\vec{g}$ are the energy flux density (Poynting vector), the energy density the electromagnetic stress tensor and the momentum density of the system, respectively. $\overrightarrow{\overrightarrow{\mathcal{I}}}$ is the unit tensor in static de Sitter space-time. Eq. (31) and Eq. (32) are the vierbein formalism of energy-momentum conservation law of the LV electrodynamics. One can observe again that these equations are different from their cousins in Lorentz invariant formalism.

As an application of the electromagnetic field equations in static de Sitter space-time, we now introduce the electrostatic field due to a point charged particle. Let's pay a little attention on a point charge (with charge q) at the point $\vec{r_0} = (r_0, \theta_0, \phi_0)$. One may ask a question: how to define a real point charge in the local frame? The answer is connected with the current conservation law in static de Sitter space-time, that is $\delta j = 0$. Using the vierbein formalism, this conservation law can be written as

$$\tilde{\nabla} \cdot (\sqrt{\sigma} \vec{j}) + \frac{\partial \rho}{\partial t} = 0. \tag{33}$$

Rewriting this equation in the form of spherical coordinate, it is

$$\nabla \cdot \tilde{\vec{j}} + \frac{1}{\sqrt{\sigma}} \frac{\partial \rho}{\partial t} = 0, \tag{34}$$

where $\tilde{\vec{j}} = (\sqrt{\sigma} j_r, j_\theta, j_\phi)$. The delta function at the point $\vec{r_0}$ in S^3 can be writen as $\delta'^3(\vec{r} - \vec{r_0}) = \sqrt{\sigma}\delta^3(\vec{r} - \vec{r_0})$, so now we can define the charge distribution function of a point charge

as $\rho = q\delta'^{\,3}(\vec{r}-\vec{r_0})$. Using the field equations we obtained above, then the electrostatic field equation becomes

$$-\tilde{\nabla}\cdot(\frac{1}{\sqrt{\sigma}}\tilde{\nabla}(\sqrt{\sigma}\varphi)) = q\delta'^{\,3}(\vec{r}-\vec{r_0}). \tag{35}$$

Utilizing the spherical coordinate, we can transform the equation to appear the formalism what we are familiar with,

$$-\nabla^2\varphi + K\frac{\partial}{\partial r}(r^2\frac{\partial\varphi}{\partial r}) + 3K\varphi + 2Kr\frac{\partial\varphi}{\partial r} + \frac{K^2r^2}{\sigma} = q\delta'^{\,3}(\vec{r}-\vec{r_0}) \tag{36}$$

However, this equation is not easy to solve. Fortunately, there is a way to round this difficulty, because the equation above is de Sitter invariance, so one can perform a suitable 'quasitranslation' in static de Sitter space-time to take the spatial origin $\mathbf{x} = \mathbf{0}$ into any $\mathbf{a}$. Then we can always choose the observed point as the origin of the local frame, namely let $r \to 0$ in Eq. (36). We then arrive at

$$-\nabla^2\varphi + 3K\varphi = q\delta^3(\vec{r}-\vec{r_0}). \tag{37}$$

This equation is very easy to solve by choosing reasonable boundary condition that $\varphi \to 0$ as $r_0 \to \infty$(Of course, in de Sitter space there is a horizon such that r_0 cannot really go to ∞. However, since the horizon radius R is very large, one can take the horizon as ∞.). We obtain

$$\varphi = \frac{q}{4\pi r_0}e^{-\sqrt{3K}\,r_0} \tag{38}$$

This electric potential damps a little faster than in Lorentz invariant electrodynamics. The electric field strength $\vec{E}$ at the observed point is

$$\vec{E} = -q\frac{\vec{r}_0}{4\pi r_0^3}e^{-\sqrt{3K}\,r_0} + q\frac{\sqrt{3K}\vec{r}_0}{4\pi r_0^2}e^{-\sqrt{3K}\,r_0}. \tag{39}$$

This formalism is obviously different from the Coulomb Theorem. Though the modification is very small, the electrostatic field strength of a point charge in our LV electrodynamics model does not exactly decay as r^{-2}. There is an another exponential damping factor in the potential, which makes the potential looks like a Yukawa one. This effect becomes important in the region of far field and it may affect the large-scale universal observation. However, since $K = \frac{1}{R^2} = \frac{\Lambda}{3}$ and R could be a very large distance parameter, say the 'radius of universe horizon', the effect the exponential damping factor can be negligible in the existing experiments.

Next we turn to focus our attention on magnetostatic field in static de Sitter space-time. The simplest and also the most fundamental case is the magnetic field of a small circle electric current. We set the center of the small circle current (with the electric current strength I and the radius a) at the point $\vec{r}_0$, and the observer is at the origin as the case of electrostatic field mentioned above. This case, however, is a little different from the electrostatic field, because the gauge condition we are apt to select is the de Sitter gauge condition Eq. (26). Using this gauge condition, one can arrive at

$$\tilde{\nabla} \cdot \vec{A} = \frac{K}{\sqrt{\sigma}} \vec{r} \cdot \vec{A}. \tag{40}$$

According to the electromagnetic field equations in static de Sitter space-time, the differential equation of $\vec{A}$ can be written as

$$\tilde{\nabla} \times (\tilde{\nabla} \times \vec{A}) = \tilde{\vec{j}}(\vec{r}) \tag{41}$$

here from Eq. (34)

$$\tilde{\vec{j}}(\vec{r}) = \sqrt{\sigma}\vec{j}(\vec{r}), \ \vec{j}(\vec{r}) = j_r \vartheta^1 + j_\theta \vartheta^2 + j_\phi \vartheta^3 \tag{42}$$

is the conserved electric current in the local frame. However, under this formalism, we don't know how to solve it. So we should rewrite this equation in spherical coordinate as we did in the case of electrostatic field of a point charge. To do this, one should look back the essential meaning of an 'ordinary' 3-d vector in a local frame, actually, a vector in S^3 space $\vec{f} = f_r \vartheta^1 + f_\theta \vartheta^2 + f_\phi \vartheta^3$ can be corresponded to a vector in 3-d Euclidean space $\vec{f} = \frac{1}{\sqrt{\sigma}} f_r \vec{e}_r + f_\theta \vec{e}_\theta + f_\phi \vec{e}_\phi$, with this corresponding and define $\vec{A}'(\vec{r}) = \frac{1}{\sqrt{\sigma}} A_r \vec{e}_r + A_\theta \vec{e}_\theta + A_\phi \vec{e}_\phi$, then one can obtain the spherical coordinate formalism of the gauge condition Eq. (40),

$$\nabla \cdot \vec{A}' = 4KrA'_r + Kr^2 A'_{r,\,r} \tag{43}$$

Then the equations of the components of $\vec{A}'$ can be derived directly from the Eq. (41)

$$-(\nabla^2 \vec{A}')_r + 4KA'_r + 6KrA'_r,\, r + Kr^2 A'_{r,r,r} = \frac{1}{\sqrt{\sigma}} j_r \tag{44}$$

$$-(\nabla^2 \vec{A}')_\theta + [K(r(rA'_\theta)_{,r})_{,r} + 3KA'_{r,\theta}] = j_\theta \tag{45}$$

$$-(\nabla^2 \vec{A}')_\phi + [K(r(rA'_\phi)_{,r})_{,r} + \frac{3}{\sin\theta} KA'_{r,\phi}] = j_\phi. \tag{46}$$

Here, however, we have no reason to say that the vector $(\frac{1}{\sqrt{\sigma}} j_r, j_\theta, j_\phi)$ is just a conservation current density in spherical coordinates. Actually, in static de Sitter space-time, the conservation current density vector must be defined from the Eq. (34). Therefore we can obtain

the current strength I through a certain cross section S' in spherical coordinates

$$I = \int (\sqrt{\sigma}\vec{j}) \cdot d\vec{S}' = \int \sqrt{\sigma} j_r r d\theta' \wedge r' \sin\theta' d\phi' + \int j_\theta dr' \wedge r' \sin\theta' d\phi' + \int j_\varphi dr' \wedge r' d\theta'. \tag{47}$$

So the conservation current density vector in spherical coordinates is $\vec{j}''(r) = \sqrt{\sigma} j_r \vec{e}_r + j_\theta \vec{e}_\theta + j_\phi \vec{e}_\phi$. Under this definition, Eqs. (44) should be multiplied a factor σ. In the limit of $r \to 0$, one can prove that a symmetric solution of Eqs.(44)-(46) is a solution of a vector equation as follows. It is easy to show that the solution of Eqs.(44)-(46) can be obtained by Bio-Savart Theorem for a small circle current placed at the origin in usual Lorentz invariant electrodynamics by setting K to 0. Then pulling the source back to $\vec{r}_0$, one can obtain A'_r, A'_θ, A'_ϕ with this solution. Suppose that the solution in the spherical frame with origin at the center of the circle is $A'_{r'}, A'_{\theta'}, A'_{\phi'}$, it is easy to show that the only non-vanishing component is $A'_{\phi'}$ and $\vec{A}'(0) = A'_{\phi'}\vec{e}_{\phi'} = -A'_{\phi'} \sin\theta \sin(\phi-\phi_0)\vec{e}_r - A'_{\phi'}\cos\theta \sin(\phi-\phi_0)\vec{e}_\theta - A'_{\phi'}\cos(\phi-\phi_0)\vec{e}_\phi$, where (r_0, θ_0, ϕ_0) is the spherical coordinates of $\vec{r}_0$. Substituting this solution to K terms in Eqs.(44)-(46) and noting that $\sigma \to 1$ as $r \to 0$, one can obtain the vector equation as follows

$$-\nabla^2 \vec{A}'(\vec{r}) + 4K\vec{A}'(\vec{r}) = \vec{j}'(\vec{r}) \tag{48}$$

This equation is invariant under a translation in parameter $\mathbf{x}$ space, so we can build a spherical coordinate frame (r', θ', ϕ') with origin at the point r_0. It is easy to show that the only non-vanishing component of the solution is still $A'_{\phi'}$ and $\vec{A}'(0) = A'_{\phi'}\vec{e}_{\phi'} = -A'_{\phi'}\sin\theta\sin(\phi-\phi_0)\vec{e}_r - A'_{\phi'}\cos\theta\sin(\phi-\phi_0)\vec{e}_\theta - A'_{\phi'}\cos(\phi-\phi_0)\vec{e}_\phi$ also holds, where

$$\vec{A}' = A'_{\phi'}\vec{e}_{\phi'} = \frac{Ia}{4\pi}\oint_0^{2\pi} \frac{\cos\varphi d\varphi}{\sqrt{r_0^2 + a^2 - 2r_0 a \sin\theta' \cos\varphi}} e^{-\sqrt{4K}\sqrt{r_0^2+a^2-2r_0 a\sin\theta'\cos\varphi}}\vec{e}_{\phi'} \tag{49}$$

In the cases $2r_0 a \sin\theta' \ll r_0^2 + a^2$, namely in the region of far field ($r_0 \gg a$), and $r_0 \sin\theta' \ll a$, the so called region of adaxial field, the above integral can be approximatively calculated to 3-order

$$\begin{aligned} A'_{\phi'} = \frac{Ia}{4\pi}\int d\varphi \cos\varphi[\mathcal{P}\frac{r_0 a \sin\theta' \cos\varphi}{(r_0^2+a^2)^{3/2}} + \mathcal{N}\frac{r_0^3 a^3 \sin^3\theta' \cos^3\varphi}{(r_0^2+a^2)^{7/2}}] \\ = \frac{Ia}{4\pi}[\mathcal{P}\frac{r_0 a \sin\theta'}{(r_0^2+a^2)^{3/2}} + \frac{3}{4}\mathcal{N}\frac{r_0^3 a^3 \sin^3\theta'}{(r_0^2+a^2)^{7/2}}], \end{aligned} \tag{50}$$

where

$$\mathcal{P} = \frac{e^{-\sqrt{4K(r_0^2+a^2)}}}{2}(1 + \sqrt{4K(r_0^2+a^2)})$$

and

$$\mathcal{N} = \frac{e^{-\sqrt{4K(r_0^2+a^2)}}}{48}\{15 + 15\sqrt{4K(r_0^2+a^2)} + 24K(r_0^2+a^2) + [\sqrt{4K(r_0^2+a^2)}\ ]^3\}.$$

Pulling the origin back to the field point by setting $\theta' = \pi - \theta_0, \phi' = \phi_0 - \pi$ then the magnetic potential $\vec{A}$ (at the origin) of the circle electric current can be written as

$$\begin{aligned} \vec{A}(0) = \vec{A}'(0) = A'_r\vec{e}_r + A'_\theta\vec{e}_\theta + A'_\phi\vec{e}_\phi &= -A'_{\phi'}\sin\theta\sin(\phi-\phi_0)\vec{e}_r \\ &-A'_{\phi'}\cos\theta\sin(\phi-\phi_0)\vec{e}_\theta - A'_{\phi'}\cos(\phi-\phi_0)\vec{e}_\phi. \end{aligned} \tag{51}$$

This solution shows that the vector potential of the circle current is also a damping potential, it decays a little faster than that in Lorentz invariant electrodynamics as the case of electrostatic field of a point charge mentioned above. So it is reasonable to say that the magnetic field strength also have a damping factor. In the region of the far field, this damping can not be ignored. In this LV electrodynamics approach, at least on very large scale observation, the LV effect become important and the observation data should be reconsidered because of the damping factor.

In conclusion, we set up an effective low energy LV classical electrodynamic model in Minkowski space-time by the covariant fomulation of electrodynamic in static de Sitter space-time. We define the observable in the model as the vierbein decomposition components of physical tensors. The electromagnetic field equations are obtained in this formalism and its deviation from ones in Lorentz invariant theory is showed. Furthermore, we investigate the energy-momentum conservation law in this LV model. As an application of the LV electromagnetic equations, we studied two basic and simple cases which might be responsible for possible observation confirmation: the electrostatic field of a point charge and the magnetostatic field of a circle electric current. We find that in both case there is an analogous damping factor in the potential function, this can be regarded as a side face of LV effect and may be important on large scale observations.

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